Kinks in the presence of rapidly varying perturbations

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The dynamics of sine-Gordon kinks in the presence of rapidly varying periodic perturbations of different physical origins is described analytically and numerically. The analytical approach is based on asymptotic expansions, and it allows one to derive, in a rigorous way, an effective nonlinear equation for the slowly varying field component in any order of the asymptotic procedure as expansions in the small parameter ω^{-1}, ω being the frequency of the rapidly varying ac driving force. Three physically important examples of such a dynamics, i.e., kinks driven by a direct or parametric ac force, and kinks on a rotating and oscillating background, are analyzed in detail. It is shown that in the main order of the asymptotic procedure the effective equation for the slowly varying field component is *a renormalized sine-Gordon equation* in the case of the direct driving force or rotating (but phase locked to an external ac force) background, and it is *the double sine-Gordon equation* for the parametric driving force. The properties of the kinks described by the renormalized nonlinear equations are analyzed, and it is demonstrated analytically and numerically which kinds of physical phenomena may be expected in dealing with the renormalized, rather than the unrenormalized, nonlinear dynamics. In particular, we predict several qualitatively new effects which include, e.g., the perturbation-induced internal oscillations of the 2π kink in a parametrically driven sine-Gordon model, and the generation of kink motion by a pure ac driving force on a rotating background.

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I. INTRODUCTION

As is well known, the effect of rapidly varying perturbations on the dynamics of nonlinear systems may lead to a drastic change of the system behavior in the sense of the *averaged* dynamics. In particular, large-amplitude *parametric* perturbations may give rise to *a stabilization* of certain types of dynamical regimes. A typical and famous example is a stabilization of a reverse pendulum by parametric forced oscillations of its pivot [1] (see also the recent paper [2] and references therein), and a similar effect may be also achieved by applying a direct ac driving force of large amplitude [3]. Such a dynamical stabilization has its analog in nonlinear systems with distributed parameters supporting, in particular, unusual types of kink solitons [4–7]. However, the method which is usually used to derive an averaged equation describing the system dynamics in the presence of rapidly varying perturbations is not rigorous and, as a matter of fact, it is not well justified. Such a method, even being very clear from the physical point of view, uses a splitting of slow and fast variables and subsequent averaging which is based, in fact, on solutions of a linearized equation for fast variations where the slowly varying coefficients are assumed to be constant [1,4,5]. The procedure of such a linearization assumes that the amplitude of the forced (rapidly varying) oscillations is small, and this is certainly valid for parametrically forced oscillations far from the parametric resonance. For direct ac perturbations, the forced oscillations may become large. To describe the dynamics in an approximate way, the so-called "rotating-wave approximation" was used without detailed mathematical justification [6–8]. It is necessary to note that the derivation of an effective averaged equation for the slowly varying field component is an important (and nontrivial) step of the analysis of such systems, and in many of the cases the corresponding equation determines the leading physical effects observed in the presence of rapidly varying perturbations. In all the cases it is necessary to justify the averaging procedure as well as to estimate the influence of the higher-order contributions. Unfortunately, the latter are beyond the usual averaging methods. However, as we show in the present paper, a rigorous analysis of the effect of rapidly varying periodic perturbations on nonlinear dynamics of ac driven damped systems can be performed in a straightforward way to describe the averaged dynamics with any accuracy.

The purpose of this paper is to present the basic steps of the method mentioned above and, selecting the sine-Gordon (SG) model as a particular example, to describe the dynamics of kinks in the presence of rapidly varying driving forces of very different physical origins. Considering the external ac driving force to be rapidly oscillating, we apply an asymptotic procedure based on a Fourier

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series where the coefficients are assumed to be slowly varying functions on the time scale ω^{-1}, ω being the frequency of the rapidly varying ac force which is assumed to be large. The basic idea to split fast and slow variables is not new, and the well-known example is, as mentioned, a stabilization of the reverse pendulum by oscillations of its suspension point. However, our analytical method to derive an effective equation for the slowly varying field component is not standard, and this method allows us to calculate, in a self-consistent way, all the corrections using solely asymptotic expansions rather than direct averaging in fast oscillations. It is clear that the applicability of the method itself is much wider than the particular examples covered by the present paper.

The paper is organized as follows. In Sec. II we consider the case of a direct ac driving force demonstrating the basic steps of our analytical approach in detail. The main result of such an analysis is the so-called averaged equation, i.e., that describing slowly varying system dynamics. In the case of the direct driving force this equation is shown to be a renormalized SG equation. Section III presents the case of a parametric driving force where the final equation describing the slowly varying dynamics is the double SG equation which, as we show, may display new features in the averaged kink dynamics, e.g., oscillations of the excited internal mode of the kink, which is absent in the standard SG model. The case of kink stabilization on a rotating background by applying a rapidly oscillating ac force is discussed in Sec. IV. There we analyze an effect of an induced dc force on the kink motion, as well as showing numerically that the main conclusions of our analysis may be easily extended to cover multisoliton dynamics. Finally, Sec. V concludes the paper.

II. DIRECT DRIVING FORCE

A. Asymptotic expansions

As the first example, we consider the case of the direct driving force in the SG model when the system dynamics is described by the driven damped SG equation for the field variable $\phi(x,t)$,

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \sin\phi = f - \gamma\frac{\partial\phi}{\partial t} + \epsilon\cos(\omega t), \tag{1}$$

where f is a constant contribution of the driving force, γ is the damping coefficient, and the amplitude ϵ of the driving force may be large (in fact, up to the values of order of ω^2). The standard physical application of the model (1) is to describe the fluxon dynamics in long Josephson junctions (see, e.g., [9]), so that f and $\epsilon\cos(\omega t)$ are the constant and varying components of the bias current applied to the junction. In the subsequent analysis we consider the direct driving force ($\sim\epsilon$) as *rapidly oscillating*, i.e., the frequency ω is assumed to be large in comparison with the frequency gap ($=1$) of the linear spectrum band. Our purpose is to derive an averaged nonlinear equation to describe the *slowly varying* dynamics of the SG field.

In order to derive an averaged equation of motion, we note first that in the case of very different time scales the SG field ϕ may be decomposed into a sum of slowly and rapidly varying parts, i.e.,

$$\phi = \Phi + \zeta. \tag{2}$$

The function ζ stands for fast oscillations around the slowly varying envelope function Φ, and the mean value of ζ during an oscillation period is assumed to be zero so that $\langle\phi\rangle = \Phi$. Our goal is to derive an effective equation for the function Φ. The standard way to do that is to substitute Eq. (2) into Eq. (1) and to split Eq. (1) into two equations for slow and fast variables, making an averaging to obtain the equation for the slowly varying field component (see, e.g., [4,5]). However, such an approach must be properly justified for the case when the fast oscillations *are not small* as it is for the direct driving force considered here, and in a similar problem it was proposed [6] to use the so-called rotating-wave approximation to find the rapidly oscillating field component. All these approaches, although quite satisfactory for the first-order approximation (see, e.g., [4,6,7]), do not allow to make the next-order expansions to calculate higher-order corrections, and thus they cannot be rigorously justified. Nevertheless, as we show in the present paper, a rigorous approach may indeed be proposed to obtain an effective equation for the slowly varying field component Φ *with any accuracy.*

The basis of our asymptotic procedure is a Fourier series expansion with slowly varying coefficients. We look for rapidly oscillating component ζ in the form

$$\zeta = A\cos(\omega t) + B\sin(\omega t) + C\cos(2\omega t) + D\sin(2\omega t) + \cdots, \tag{3}$$

where the coefficients A, B, ... are assumed to be slowly varying on the time scale $\sim\omega^{-1}$. Substituting the expressions (2) and (3) into Eq. (1), we note that the effective coupling between different harmonics of the expansion (2) and (3) is produced by the nonlinear term $\sin\phi$, which generates the following Fourier expansion:

$$\sin\phi = \sin\Phi\,[\alpha_0 + \alpha_1\cos(\omega t) + \alpha_2\sin(\omega t) + \alpha_3\cos(2\omega t) + \alpha_4\sin(2\omega t) + \ldots] + \cos\Phi\,[\beta_0 + \beta_1\cos(\omega t) + \beta_2\sin(\omega t) + \beta_3\cos(2\omega t) + \beta_4\sin(2\omega t) + \ldots], \tag{4}$$

where

$$\alpha_0 = J_0(A)\left(1-\frac{1}{4}B^2\right)+\frac{1}{4}B^2J_2(A)+\cdots,$$
$$\alpha_1 = -CJ_1(A)+\cdots,$$
$$\alpha_2 = -DJ_1(A)+\cdots, \alpha_3 = \frac{1}{4}B^2J_0(A)+\cdots,$$
$$\alpha_4 = -BJ_1(A)+\cdots; \quad (5)$$

$$\beta_0 = -CJ_2(A)+\cdots, \quad \beta_1 = 2J_1(A)+\cdots,$$
$$\beta_2 = BJ_0(A)+BCJ_1(A)+\cdots,$$
$$\beta_3 = CJ_0(A)+\cdots, \quad \beta_4 = DJ_0(A)+\cdots; \quad (6)$$

and J_0, J_1, etc., are Bessel functions. Collecting now the coefficients in front of the different harmonics, we obtain an infinite set of coupled nonlinear equations,

$$\frac{\partial^2\Phi}{\partial t^2}-\frac{\partial^2\Phi}{\partial x^2}+\sin\Phi\left[J_0(A)\left(1-\frac{1}{4}B^2\right)+\frac{1}{4}B^2J_2(A)+\cdots\right]+\cos\Phi\left[-CJ_2(A)+\cdots\right]=f-\gamma\frac{\partial\Phi}{\partial t}, \quad (7)$$

$$\left(-\omega^2A+2\omega\frac{\partial B}{\partial t}+\frac{\partial^2A}{\partial t^2}\right)-\frac{\partial^2A}{\partial x^2}+\cos\Phi[2J_1(A)+\cdots]+\sin\Phi[-CJ_1(A)+\cdots]+\gamma\left(\frac{\partial A}{\partial t}+\omega B\right)=\epsilon, \quad (8)$$

$$\left(-\omega^2B-2\omega\frac{\partial A}{\partial t}+\frac{\partial^2B}{\partial t^2}\right)-\frac{\partial^2B}{\partial x^2}+\cos\Phi[BJ_0(A)+BCJ_1(A)+\cdots]+\sin\Phi[-DJ_1(A)+\cdots]+\gamma\left(\frac{\partial B}{\partial t}-\omega A\right)=0, \quad (9)$$

$$\left(-4\omega^2C+4\omega\frac{\partial D}{\partial t}+\frac{\partial^2C}{\partial t^2}\right)-\frac{\partial^2C}{\partial x^2}+\cos\Phi[CJ_0(A)+\cdots]+\sin\Phi\left[\frac{1}{4}B^2J_0(A)+\cdots\right]+\gamma\left(\frac{\partial C}{\partial t}+2\omega D\right)=0, \quad (10)$$

$$\left(-4\omega^2D-4\omega\frac{\partial C}{\partial t}+\frac{\partial^2D}{\partial t^2}\right)-\frac{\partial^2D}{\partial x^2}+\cos\Phi[DJ_0(A)+\cdots]+\sin\Phi[-BJ_1(A)+\cdots]+\gamma\left(\frac{\partial D}{\partial t}-2\omega C\right)=0, \quad (11)$$

and similar equations for the coefficients in front of the higher-order harmonics. To proceed further, we note that different terms in Eqs. (8)–(11) are not equivalent provided ω is a large parameter. Indeed, if we assume the amplitude ϵ large as well (otherwise, the dynamics of the system is rather trivial because the effect of small amplitude but rapidly oscillating force is negligible), let us say up to the order of ω^2, the large term $-\omega^2A$ in Eq. (8) may be compensated only by the term ϵ from the right-hand side of Eq. (8). Thus, assuming $\epsilon \sim \omega^2$ we find the first term of the asymptotic expansion $A \approx -\epsilon/\omega^2$. On the other hand, the right-hand side of Eq. (9) is zero, and the large term $-\omega^2B$ may be compensated only by a contribution from the other terms $\sim A$, thus giving the first term of the expansion for the coefficient B, viz., $B \approx \gamma\epsilon/\omega^3$. Such a simple reasoning may be effectively applied to other coefficients as well as to other corrections of the asymptotic expansion. As a matter of fact, to generalize and simplify the procedure of calculation of the expansion coefficients, we look for the coefficients $A, B, \ldots$ in the form of the power series in the small parameter ω^{-1} as follows:

$$A = a_1+\frac{a_2}{\omega^2}+\cdots, B=\frac{b_1}{\omega}+\frac{b_2}{\omega^3}+\cdots,$$
$$C=\frac{c_1}{\omega^4}+\frac{c_2}{\omega^6}+\cdots, \quad D=\frac{d_1}{\omega^3}+\frac{d_2}{\omega^5}+\cdots. \quad (12)$$

Substituting Eq. (12) into Eqs. (8)–(11) and equating the terms of the same orders in the small parameter ω^{-1}, we find

$$a_1 = -\frac{\epsilon}{\omega^2} \equiv -\delta, \quad (13)$$

$$a_2 = \gamma b_1 + 2\cos\Phi\, J_1(a_1), \quad (14)$$

$$b_1 = -\gamma a_1, \quad (15)$$

$$b_2 = -2\frac{\partial a_2}{\partial t}-\gamma a_2+b_1\cos\Phi[J_0(a_1)+2J_2(a_1)], \quad (16)$$

$$c_1 = \frac{1}{4}\left[4\frac{\partial d_1}{\partial t}+\frac{1}{4}b_1^2J_0(a_1)\sin\Phi+\frac{1}{2}b_1^2J_2(a_1)+2\gamma d_1\right], \quad (17)$$

$$d_1 = -\frac{1}{4}b_1J_1(a_1)\sin\Phi, \quad (18)$$

and so on. In Eq. (13) the parameter $\delta = \epsilon/\omega^2$ is assumed to be of order of $\mathcal{O}(1)$, but all the results are valid also for the case $\delta \ll 1$. The expansions (12) allow us to find the coefficients $A, B, \ldots$ in each order of ω^{-1}, and all the corrections are determined by *algebraic* relations rather than additional differential equations. For example, a_2 is determined by Eq. (14) through b_1 which, in its turn, may be found from Eq. (15) as a function of a_1, i.e., through the slowly varying part Φ, and so on. This statement is valid for all coefficients of the asymptotic expansion: The coefficients are found through *algebraic* relations involving lower-order terms of the asymptotic expansions and their derivatives, and one does not need to find solutions of additional differential equations.

B. Renormalized equation

Applying the expansions (12) to Eq. (7), we may find the equation for the slowly varying field component Φ with any accuracy in the small parameter ω^{-1}, e.g.,

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + \sin\Phi\left[J_0(a_1) + \frac{1}{\omega^2}\left(-\frac{1}{4}b_1^2 J_0(a_1) - a_2 J_1(a_1) + \frac{1}{4}b_1^2 J_2(a_1)\right) + \cdots\right] + \cos\Phi\left[-\frac{c_1}{\omega^4}J_2(a_1) + \cdots\right] = f - \gamma\frac{\partial\Phi}{\partial t}. \quad (19)$$

Thus, from the asymptotic procedure described above it is quite obvious how to calculate the corrections of the first, second, and subsequent orders and to find the averaged equation with any required accuracy.

In the first-order approximation in ω^{-1} only the term $J_0(a_1)\sin\Phi$ contributes, so that Eq. (19) yields

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + J_0\left(\frac{\epsilon}{\omega^2}\right)\sin\Phi = f - \gamma\frac{\partial\Phi}{\partial t}. \quad (20)$$

Equation (20) takes into account an effective contribution of the rapidly varying force to the average nonlinear dynamics and this contribution might become large for $\delta = \mathcal{O}(1)$, i.e., when $\epsilon \sim \omega^2$. Thus the dynamics of the SG model with a rapidly varying direct driving force may be described by a *renormalized* SG equation (20) up to the terms of order of ϵ/ω^2.

The results obtained above may immediately be applied to describe the renormalized dynamics of kinks in the presence of the rapidly varying ac force. In fact, Eq. (20) is the dc driven damped SG equation with a *renormalized* coefficient in front of the term $\sim \sin\Phi$. This simply means that we can apply all the results known for the standard SG equation (see, e.g., [9,10]) making only a *renormalization* of the kink's width. For example, the kink solution of Eq. (20) at $\gamma = f = 0$ has the form

$$\Phi(x,t) = 4\sigma\tan^{-1}\exp\left[\frac{x - Vt}{l_0\sqrt{1-V^2}}\right], \quad (21)$$

where $\sigma = \pm 1$ is the kink's polarity and $l_0 = [J_0(\epsilon/\omega^2)]^{-1/2}$ is the kink's width at rest. The motion of the kink in the presence of small dc force f and damping ($\sim \gamma$) is characterized by the steady-state velocity

$$V_* = -\frac{\sigma}{\sqrt{1+g^2}}, \quad g \equiv \left(\frac{4\gamma}{\pi f}\right) J_0\left(\frac{\epsilon}{\omega^2}\right). \quad (22)$$

In the theory of long Josephson junctions the kink's velocity is connected with the voltage across the junction $\langle\phi_t\rangle$, where $\langle\,\rangle$ stands for the averaging in time, so that the result (22) for the steady-state kink velocity gives the so-called zero-field steps in the current-voltage (IV) characteristics of a long junction. As follows from Eq. (22), the renormalization of the parameter g leads to a change of the kink's velocity $V_*(f)$ and this, therefore, changes the slopes of the voltage steps by the effect of the ac driving force.

III. PARAMETRIC DRIVING FORCE

Let us consider now a parametric driving force applied to the SG system, with the main purpose to demonstrate that such a case is *very different* from that analyzed above. The qualitative difference between the effects produced by direct and parametric (rapidly oscillating) forces is the following: A sufficient change of the system averaged dynamics due to a rapidly oscillating direct force may be observed for amplitudes $\epsilon \sim \omega^2$ whereas in the case of a parametric force, similar effects may be already observed for *smaller* amplitude, i.e., in fact for $\epsilon \sim \omega$. To prove this statement and to show how our asymptotic method works for the case of the parametric force, we consider the parametrically perturbed SG equation in the form

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \sin\phi = f - \gamma\frac{\partial\phi}{\partial t} + \epsilon\sin\phi\cos(\omega t), \quad (23)$$

where f and γ have the same sense as above, but this time ϵ is the amplitude of the parametric force. Various applications of the model (23) were discussed in the review paper [10] (see also Ref. [11]).

We assume that the parametric force is rapidly oscillating, i.e., the frequency ω is large. As above, we look for a solution of Eq. (23) in the form of asymptotic expansion

$$\phi = \Phi + A\cos(\omega t) + B\sin(\omega t) + C\cos(2\omega t) + D\sin(2\omega t) + \cdots, \quad (24)$$

where the functions $\Phi, A, B, \ldots$ are assumed to be slowly varying on the time scale $\sim \omega^{-1}$. The function Φ in Eq. (24) determines, in fact, the evolution of the averaged field component because $\langle\phi\rangle = \Phi$, where the brackets $\langle\,\rangle$ stand for the averaging in fast oscillations. Substituting the expression (24) into Eq. (23) and collecting, as in the case of the direct driving force, all the coefficients in front of the different harmonics, we again obtain an infinite set of coupled nonlinear equations. The subsequent (and very important) step of such an analysis is to find the form of the asymptotic expansions for the coefficients $A, B, \ldots$. In the present case it is easy to check that the expansions (12) do not give a closed asymptotic procedure, and in the case $\epsilon \sim \omega^2$ the driving force from Eq. (23) contributes to all the harmonics, so that contributions of the other harmonics become large as well. Comparing, as in the previous case, different terms of the equations for the coefficients $A, B, \ldots$, we may easily check that the asymptotic procedure may be effectively formulated for *smaller* (but not small) amplitudes, i.e., when $\epsilon \sim \omega$, and, as above, it gives all the corrections to the averaged nonlinear dynamics in a rigorous way. Thus we take the asymptotic expansions in the form

$$A = \frac{a_1}{\omega^2} + \frac{a_2}{\omega^4} + \cdots, \quad B = \frac{b_1}{\omega^3} + \frac{b_2}{\omega^5} + \cdots,$$
$$C = \frac{c_1}{\omega^4} + \cdots, \quad D = \frac{d_1}{\omega^5} + \cdots . \qquad (25)$$

Using the power series (25), we obtain the "averaged" equation in the form

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \left(1 - \frac{1}{4}\frac{a_1^2}{\omega^4} + \cdots\right) \sin \Phi$$
$$= f - \gamma \frac{\partial \Phi}{\partial t} + \frac{\epsilon}{2} \cos \Phi \left(\frac{a_1}{\omega^2} + \frac{a_2}{\omega^4} + \cdots\right), \qquad (26)$$

The expansions (25) allow to find the coefficients of the asymptotic expansions in each order in the small parameter ω^{-1}, and all the corrections are determined, as above, by *algebraic* relations.

In the first-order approximation only the term $\sim \epsilon a_1$ contributes to Eq. (26). From the asymptotic expansions it follows that

$$a_1 = -\epsilon \sin \Phi, \qquad (27)$$

and Eq. (26) yields

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \left(1 + \frac{1}{2}\Delta^2 \cos \Phi\right) \sin \Phi = f - \gamma \frac{\partial \Phi}{\partial t}, \qquad (28)$$

where $\Delta \equiv \epsilon/\omega$. Equation (28) takes into account an effective contribution of the rapidly varying parametric force to the slowly varying nonlinear dynamics *in the lowest order*, and all the corrections coming from the approximation of the next order are proportional to the small parameter $\sim \omega^{-2}$. However, even the lowest-order contribution might become large for $\Delta = \mathcal{O}(1)$, i.e., when $\epsilon \sim \omega$.

Thus the averaged dynamics of the SG model with a rapidly varying parametric force is described by the double SG equation (28). As a matter of fact, the double SG equation is rather well studied (see, e.g., Refs. [12,13] and references therein) and properties of its kink solutions are known as well. In particular, the kink solution of Eq. (28) at $f = \gamma = 0$ may be written in the form [12]

$$\Phi(x,t) = 2\tan^{-1}\left[\frac{1}{\sqrt{1+\Delta^2/2}} \times \operatorname{csch}\left(\sqrt{1+\frac{\Delta^2}{2}}\frac{x-Vt}{\sqrt{1-V^2}}\right)\right], \qquad (29)$$

and this solution may be treated as two coupled π kinks. In Fig. 1 we show the results of numerical simulations of the parametrically driven SG system, Eq. (23). In all the cases analyzed in the present paper we have integrated the driven damped SG equation on a spatial interval of length L, with periodic boundary conditions. As seen from Fig. 1, the sech-type shape of the 2π kink corresponding to the standard (unperturbed) SG system is modified, and the function ϕ_x displays a two-peaked profile which, as a matter of fact, is one of the main fea-

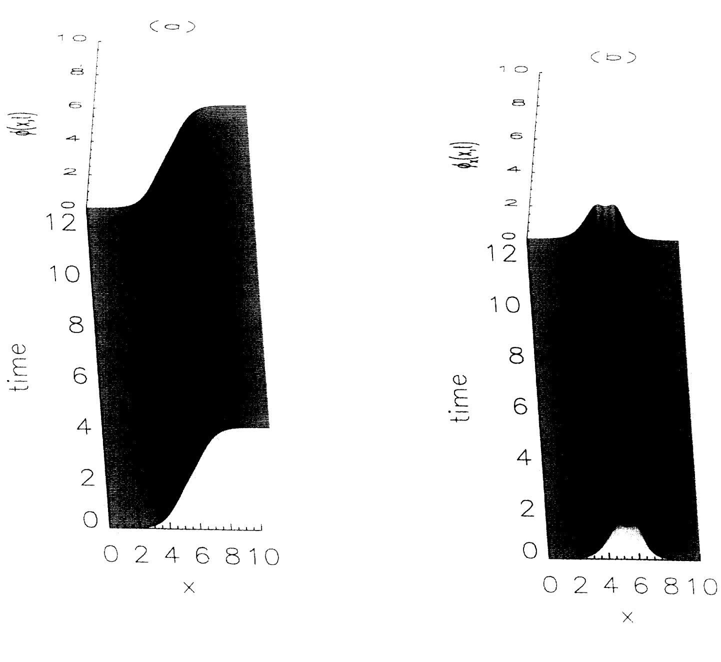

FIG. 1. The steady-state profile of the (a) ϕ and (b) ϕ_x fields as a function of space in the case of a parametric driving force. The parameters are $\gamma = 0.2$, $L = 10$, $\omega = 100$, and $\epsilon = 200$.

tures of the kink solution (29). Increasing Δ^2 one may observe, in accordance with Eq. (29), that the function ϕ has an evident shape of two π kinks separated by a distance $\sim \Delta^2$.

As has been shown in Ref. [4], π kinks themselves may exist in the parametrically driven SG chain provided the condition $\Delta^2 > 2$ is satisfied. This condition simply means that the effective averaged potential for the slowly varying field component Φ exhibits a local minimum at $\Phi = \pi$ so that this stationary state becomes stable.

The appearance of new features in the slowly varying (averaged) system dynamics of the SG system for $\Delta^2 > 2$ is similar to the phenomenon of the parametric stabilization of the reverse pendulum in the well known Kapitza problem [1,2]. However, in the problem under consideration some interesting features in the nonlinear dynamics of the parametrically driven SG system may be really observed for *any value of the effective parameter* Δ^2. Indeed, as is known from the theory of the double SG equation [13], at any value of Δ^2 the kink (29) possesses the so-called internal ("shape") mode which describes variations of the kink's width. This internal mode is absent for the standard SG kink, and the mode frequency $\Omega^2_{\rm sol}$ splits at any $\Delta^2 \neq 0$ from the gap frequency of the linear spectrum. For $\Delta^2 > 2$ the kink's internal mode may be described as relative oscillations of the π kinks of which the 2π kink consists. However, this mode does, in fact, exist at any value of the effective parameter Δ^2, and it may be observed as periodic variations of the kink's shape.

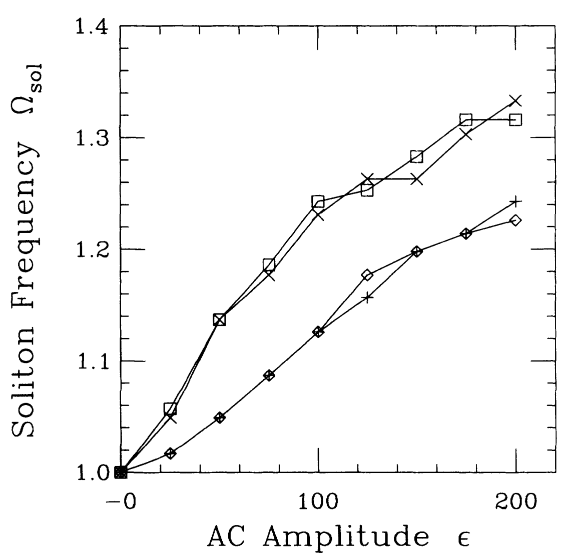

FIG. 2. The frequency of the internal oscillations of the 2π kink, $\Omega_{\rm sol}$, as a function of the amplitude of the ac parametric force ϵ for two values of the external force frequency $\omega = 50$ (squares and crosses) and $\omega = 100$ (diamonds and pluses). The results presented by diamonds and squares are obtained using the effective double SG equation (28) whereas the pluses and crosses are the result of direct integration of the parametrically driven SG equation (23). Note that the agreement between the parametrically driven model (23) and the effective ("averaged") model (28) is better for smaller $\Delta = \epsilon/\omega$, when corrections to Eq. (28) from the higher-order terms are negligible.

We have measured numerically the frequency of the shape oscillations of the 2π kink (29) directly solving Eq. (23) and also using the averaged equation (28). The numerical results are shown in Fig. 2 for selected values of the external frequency, $\omega = 50$ and $\omega = 100$. For relatively small Δ^2 (i.e., ϵ in Fig. 2), when higher-order corrections to Eq. (28) are negligible, a perfect agreement between the results for the parametrically driven SG model (23) and those for the averaged equation (28) are clearly observed, justifying the validity of our asymptotic procedure.

IV. KINKS ON ROTATING AND OSCILLATING BACKGROUNDS

As was mentioned in Ref. [7], the other physically important case when a rapidly varying ac force may change drastically the kink dynamics is the case of a rotating and oscillating background. We should note, however, that if one considers relatively small system's length L, even a relatively weak driving force may lead to complicated dynamics involving coexisting states of bunched kinks and nontrivial background states [14]. These latter effects are probably caused by the influence of nonzero boundary conditions which may "help to lock" kinklike states on rotating backgrounds. Here we are interested in the dynamics of the long SG systems (i.e., the kink's length is much smaller than the system length L) when the high-frequency force phase locks the SG field in an oscillating and rotating state and thereby creates a mechanism (an effective gravitation field) for supporting kink solitons. However, we should note that the theory presented below cannot be applied to formally infinite SG system because for the case of the continum linear spectrum the applied ac force may create resonances making the system dynamics much more complicated and even chaotic. In fact, we need finite-width (but of large L) systems in order to avoid linear resonances if the frequency of the external ac force is selected in a gap between the nearest eigenfrequencies.

To describe the effect of the kink phase locking on a rotating background analytically in a rigorous way, we consider the perturbed SG equation (1) assuming $f > 1$, in which case the ground state of the SG chain is not stable and the chain rotates with the frequency Ω so that $\phi \approx \Omega t$. Applying the high-frequency ac force $\sim \epsilon$ we are interested in the slowly varying phase locked system dynamics on such a rotating background. Accordingly, we look for a solution of Eq. (1) in the form

$$\phi = \Phi + \Omega t + \xi, \qquad (30)$$

where ξ is the rapidly varying part oscillating with the large frequency ω of the external ac force, Φ is the slowly varying (long time scale) part, and Ω is the average frequency of rotation for the background field, which we assume to be phase locked to the external ac field, i.e. $\Omega = \pm k\omega$, k being integer. Looking for the rapidly oscil-

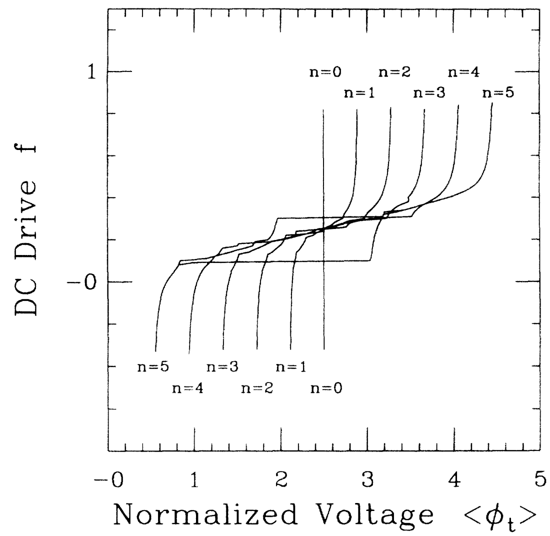

FIG. 3. The normalized IV curves (for the first Shapiro step) characterizing the SG dynamics with the parameters $\gamma = 0.1$, $L = 16$, $\epsilon = 12.5$, and $\Omega = 2.5$ and with periodic boundary conditions. Shown are the zero-field steps at $n = 0, 1, 2, 3, 4$, and 5, n being the number of kinks in the system. Note that the IV curves clearly cross the zero current axis and the steps are slightly asymmetric around the voltage $\langle\phi_t\rangle = 2.5$; the latter effect is caused by the background oscillations and relatively small velocity.

lating part ξ in the form of a Fourier series with slowly varying coefficients,

$$\xi = A\cos(\omega t) + B\sin(\omega t) + \cdots, \tag{31}$$

we obtain the following equations for the averaged field component Φ and the expansion coefficients $A, B, \ldots$,

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + J_k(A)\sin\Phi = f - \gamma k\omega - \gamma\frac{\partial\Phi}{\partial t}, \tag{32}$$

$$\left(-\omega^2 A - 2\omega\frac{\partial B}{\partial t} + \frac{\partial^2 A}{\partial t^2}\right) - \frac{\partial^2 A}{\partial x^2} + [J_{k+1}(A) - J_{k-1}(A)]\cos\Phi + \gamma\left(\frac{\partial A}{\partial t} + \omega B\right) = \epsilon, \tag{33}$$

$$\left(-\omega^2 B + 2\omega\frac{\partial A}{\partial t} + \frac{\partial^2 B}{\partial x^2}\right) - \frac{\partial^2 B}{\partial x^2} - [J_{k+1}(A) + J_{k-1}(A)]\sin\Phi + \gamma\left(\frac{\partial B}{\partial t} - \omega A\right) = 0, \tag{34}$$

and so on. Unlike the case considered in Sec. II, in the present problem there are two rapidly oscillating contributions with the frequencies ω and $k\omega$, so that the final equations (32)–(34) for the slowly varying coefficients differ from the corresponding equations (7)–(9) of Sec. II. To take into account the second term in the right-hand side of Eq. (32) in a self-consistent way, we also assume the dissipation to be rather small, $\gamma\omega \sim 1$, which is a typical case for Josephson junctions.

Making now asymptotic expansions similar to the case of the direct ac force considered above, i.e.,

$$A = a_1 + \frac{a_2}{\omega^2} + \cdots, \quad B = \frac{b_1}{\omega} + \cdots, \tag{35}$$

we find the following relations [cf. Eqs. (13) and (15)]:

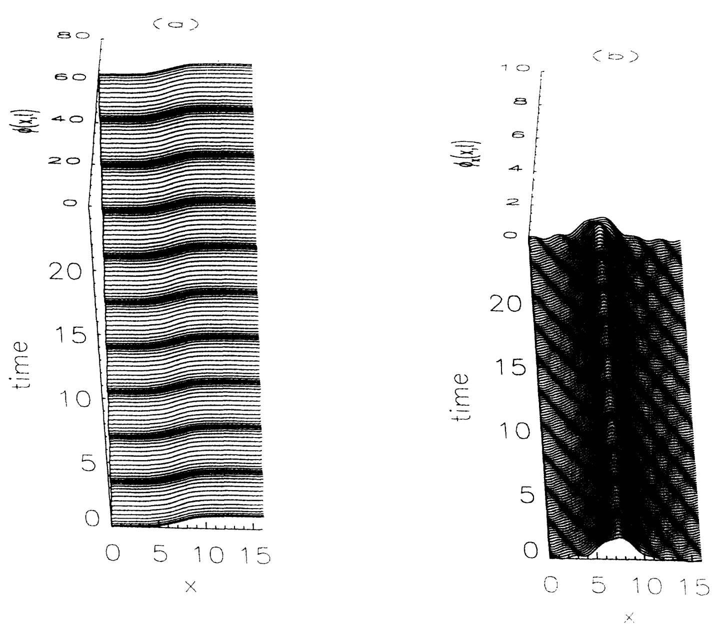

FIG. 4. The steady-state profile of the (a) ϕ and (b) ϕ_x fields as a function of space over ten periods of the external driving force. Parameters are $L = 16$, $\gamma = 0.1$, $\epsilon = 12.5$, $\omega = 2.5$, and $f = 0.25$. The value of f is selected at the center of the first Shapiro step and therefore the kink does not move as follows from the theory because the effective force acting on a kink $f - \gamma k\omega$ is zero.

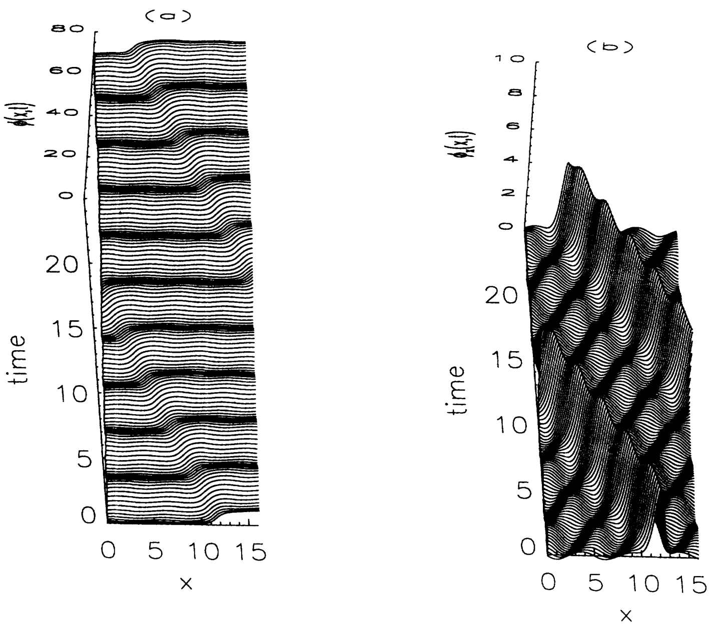

FIG. 5. The same as in Fig. 4 but at $f = 0.5$.

$$a_1 = -\frac{\epsilon}{\omega^2} \equiv -\delta, \quad b_1 = -\gamma a_1, \tag{36}$$

which allow us to obtain the effective equation for the slowly varying system dynamics by just combining Eqs. (36), (35), and (32). The final equation is just Eq. (32) with $A = -\epsilon/\omega^2$, which describes the kink dynamics on the background rotating with the frequency $\Omega = \pm k\omega$. It is important to note that the resulting (effective) dc force in the averaged nonlinear equation (32) is represented by the term $f - \gamma k\omega$ but not f itself, i.e., the kink on the rotating and oscillating background may move even in the absence of the constant contribution to the bias current $f = 0$. Figure 3 shows the results of the numerical calculation of the first Shapiro step ($k = 1$) of a long Josephson junction described by the model (1) with the parameters $\gamma = 0.1$, $\epsilon = 12.5$, and $\Omega = \omega = 2.5$. As is clearly seen from that figure, the steps cross the zero current axis, displaying the property mentioned above; constant-voltage zero-crossing steps are of considerable practical interest for voltage-standard applications of Josephson junctions (see, e.g., [15] and references therein). If we select $f = 0.25$, the effective force acting on the kink vanishes and the kink is observed at rest (see Fig. 4). This result is in excellent agreement with the averaged SG equation (32), where the effective force is found to be $f - \gamma k\omega$. Increasing the value of the bias current f, we create an effective force acting on the kink and it moves to the left (see Fig. 5). It is very interesting to note that the kink motion is even possible in the absence of the constant component of the bias current, i.e., at $f = 0$.

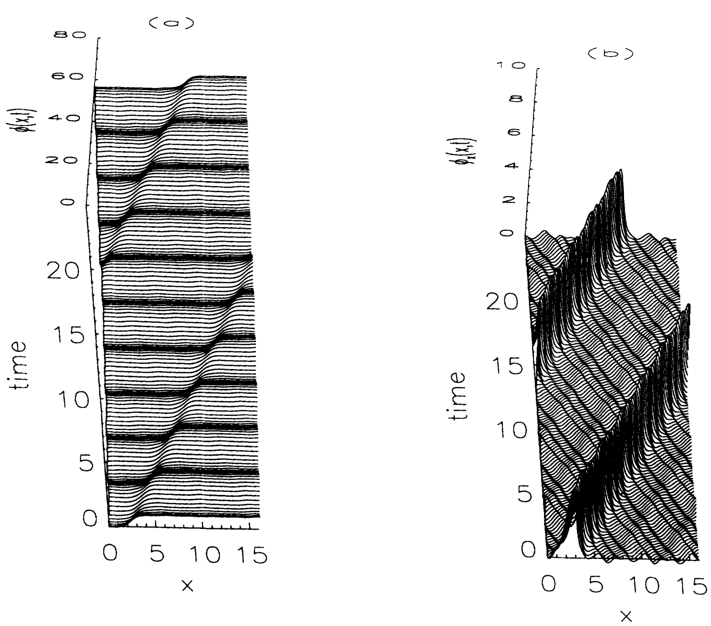

FIG. 6. The same as in Fig. 4 but at $f = 0$. The kink is moving due to the uncompensated contribution of dissipative losses $-\gamma k\omega$.

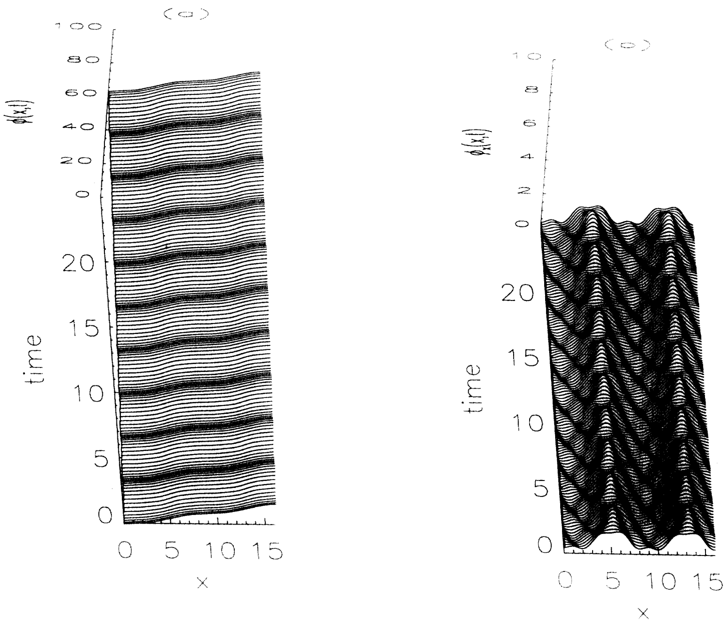

FIG. 7. The same as in Fig. 4 but for the case of two kinks introduced by a change of the boundary conditions.

Figure 6 shows this case, when the effective force $-\gamma k\omega$ generates the motion of the kink in the direction opposite to that shown in Fig. 5.

As a matter of fact, our approach and the resulting equation for the slowly varying field component Φ do not specify exactly the type of nonlinear solutions we deal with. This means that the consideration described above may be effectively applied to other problems, much more general than a single-kink propagation. One of the important generalizations of this approach is to treat multisoliton (multikink) problems. In particular, for the nonlinear dynamics on oscillating and rotating background the effects similar to those described for a single kink may be also observed for the case of two, three, and more kinks. In particular, Figs. 7 and 8 present results of our numerical simulations of the same effects as for the single kink, but for the case of two kinks in the directly driven SG model (1). As may be noted from these figures, the large-amplitude direct ac force generates some radiation, especially for the cases where the kinks are observed at rest, but still kinks exist as localized and well defined objects.

V. CONCLUSIONS

In conclusion, we have analyzed the dynamics of SG kinks in the presence of rapidly varying periodic perturbations of different physical nature. We have proposed a

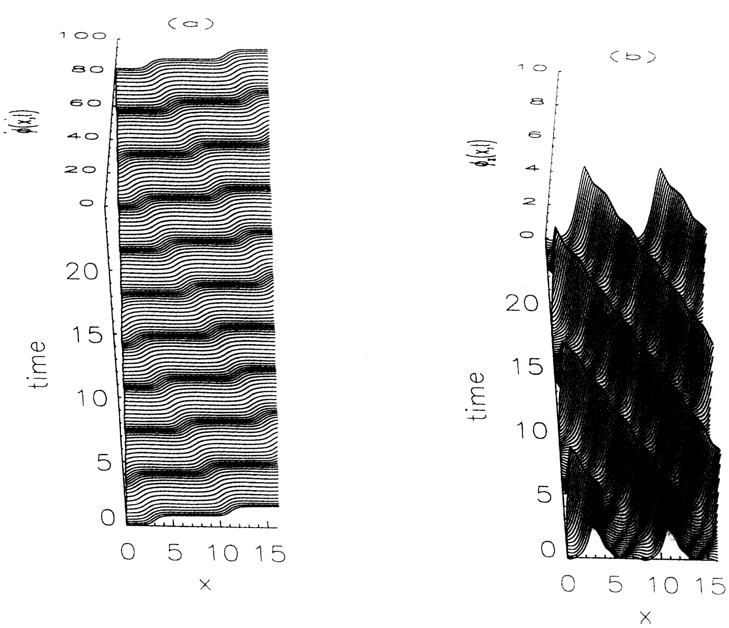

FIG. 8. The same as in Fig. 5 but for the case of two kinks.

rigorous analytical approach to derive the averaged equation for the slowly varying field component, and we have demonstrated that in the main order of the asymptotic procedure the effective equation is a renormalized SG equation in the case of the direct driving force or rotating (and phase locked to the external ac driving force) background, and it is a double SG equation for the parametric driving force. However, the method itself does allow to find in a rigorous way the effective equation for the slowly varying field component in any order of the asymptotic expansion in the parameter ω^{-1}, ω being the frequency of the rapidly varying perturbations which has been assumed to be large.

Our main purpose was to show which kinds of qualitatively new physical effects may be expected in dealing with the renormalized nonlinear dynamics instead of unrenormalized one. In particular, we have predicted that the parametric driving force may support oscillations of the kink's shape (absent in the SG model) which may be viewed as creation of a shape mode of the 2π kink characterized by the internal frequency $\Omega_{\rm sol}$. For the problem of the kink propagation on rotating and oscillating background, we have shown that a periodic ac force produces a drift in the kink motion, which may be understood as an effect described by an effective dc force to the kink motion in the framework of an averaged nonlinear dynamics.

One of the main conclusions of our analysis and numerical simulations, i.e., that the averaged nonlinear dynamics is drastically modified by rapidly varying (direct or parametric) driving force but still may be effectively described by renormalized nonlinear equations, is rather general and applicable to many other nonlinear models supporting various kinds of solitons.

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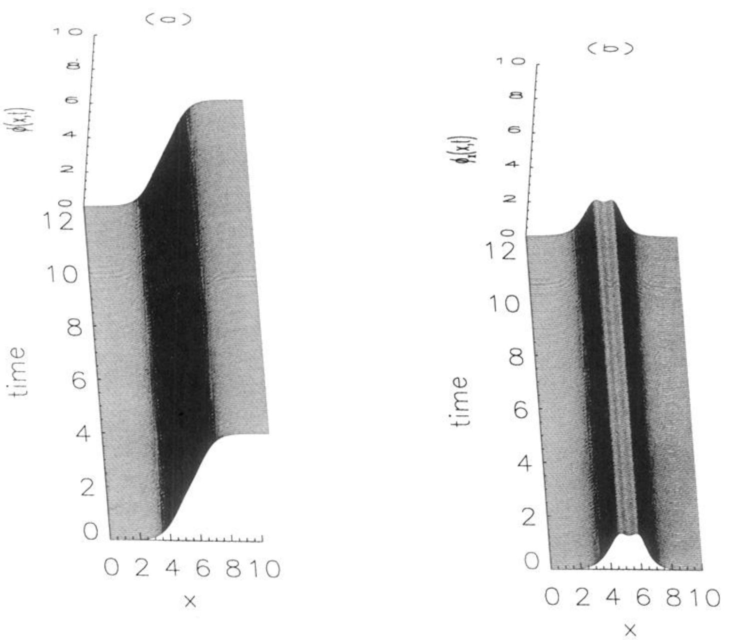

FIG. 1. The steady-state profile of the (a) ϕ and (b) ϕ_x fields as a function of space in the case of a parametric driving force. The parameters are $\gamma = 0.2$, $L = 10$, $\omega = 100$, and $\epsilon = 200$.